Dictionary-based model reduction for state estimation

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Context

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- Parametric equation in a Hilbert space U,

$$\mathcal{F}(u, \xi) = 0,$$

  $u \in U$ the **state to recover** and $\xi \in \mathcal{P} \subset \mathbb{R}^d$ an **unknown** parameter.

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Observations

of m continuous linear measurements
$\ell_1(u), \cdots, \ell_m(u),$, i.e. $w = P_W u$
with $W := \mathsf{span}\{R_U \ell_i\}_{i=1}^m$

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Observations   +   **Prior knowledge**

of m continuous linear measurements   on the solution manifold
$\ell_1(u), \cdots, \ell_m(u),$, i.e. $w = P_W u$ $\mathcal{M} := \{u(\xi) : \xi \in \mathcal{P}\}$ based
with $W := \text{span}\{R_U \ell_i\}_{i=1}^m$ model reduction

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$$\downarrow$$

Compute a recovery $A(w) \simeq u$

One space problem
Linear MOR

One space: formulation

- Approximate $\mathcal{M}$ by a linear subspace V,

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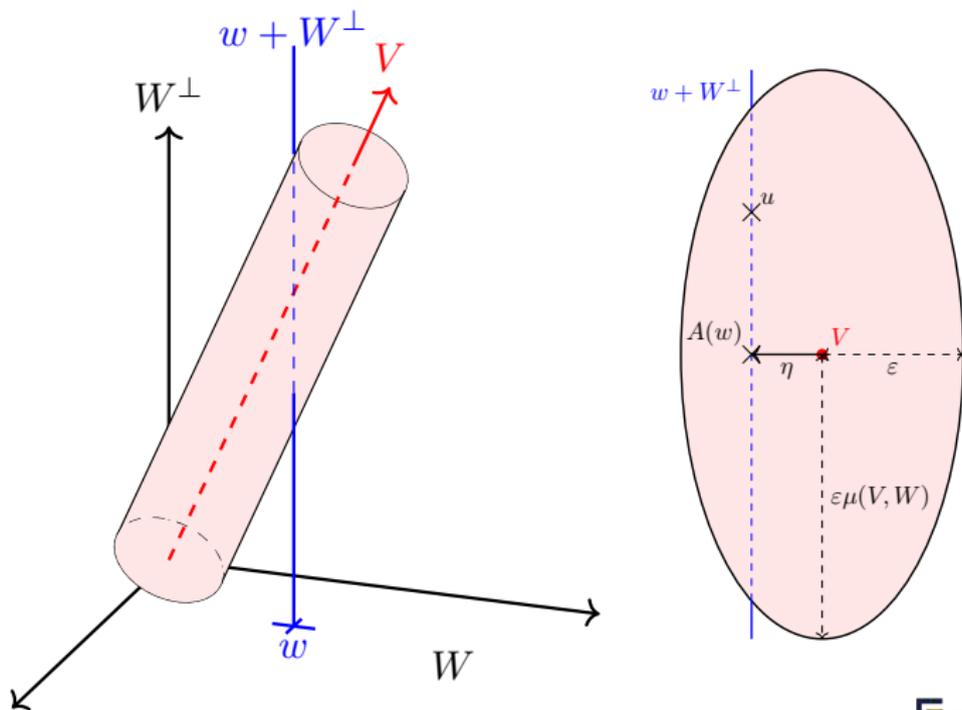$$\text{dist}(V, \mathcal{M}) \leq \varepsilon, \qquad \text{dim}(V) = n$$

- Parameterized-Background Data-Weak (PBDW) [1]:

$$A_V(w) := v^* + \eta^*, \quad v^* := \arg\min_{v \in V} \|P_W(u-v)\|, \quad \eta^* := P_W(u-v^*)$$

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- Sharp error bound [2]: $\|u - A_V(w)\| \leq \mu(V, W)\varepsilon$

$$\mu(V, W) := \beta(V, W)^{-1}, \quad \beta(V, W) := \inf_{v \in V} \sup_{w \in W} \frac{\langle v, w \rangle}{\|v\| \|w\|}$$

One space: Geometry

When $\mathcal{M}$ is the cylinder, A_V is the optimal mapping [2].

One space: Pros and cons

Pros

- Online efficiency $\mathcal{O}(n^3)$.

- No need to know ε

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- No need to know ε

Cons

- Requires $n \leq m$.

- Trade-off between ε and $\mu(V, W)$.

- Limited by the Kolmogorov m-width,

$$\varepsilon \geq d_m(\mathcal{M}).$$

Multi-space problem
Library-based MOR

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Multi-space: library-based MOR

• Consider a library of spaces $\mathcal{L}_n^N := \{V_1, \cdots, V_N\}$

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- **Goal**: Select a good one.

Multi-space: How ?

How to select k^*
among this library ?

Multi-space: Selection

Idea [3]: select the "closest" to $\mathcal{M}$

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- Assume we have $\mathcal{S}$ such that

$$c \text{ dist}(\cdot, \mathcal{M}) \leq \mathcal{S}(\cdot, \mathcal{M}) \leq C \text{ dist}(\cdot, \mathcal{M})$$

- Select k^* as

$$k^* \in \underset{1 \leq k \leq N}{\operatorname{argmin}} \ \mathcal{S}(A_k(w), \mathcal{M}).$$

Multi-space: Selection

Proposition (Near optimal selection [3])

Assuming that P_W is injective on $\mathcal{M}$ and that $\mu(\mathcal{M}, W) < \infty$,

$$\|u - A_{k^*}(w)\| \le 2\frac{C}{c}\mu(\mathcal{M}, W) \min_{1 \le k \le N} \|u - A_k(w)\|,$$

$\mu(\mathcal{M}, W)$ reflects how well $\mathcal{M}$ and W are aligned.

Dictionary-based multi-space
Dictionary-based MOR

Dictionary multi-space: Library

• Dictionary of K vectors (or snapshots),

$$\mathcal{D}_K = \{v^{(1)}, \cdots, v^{(K)}\}$$

Dictionary multi-space: Library

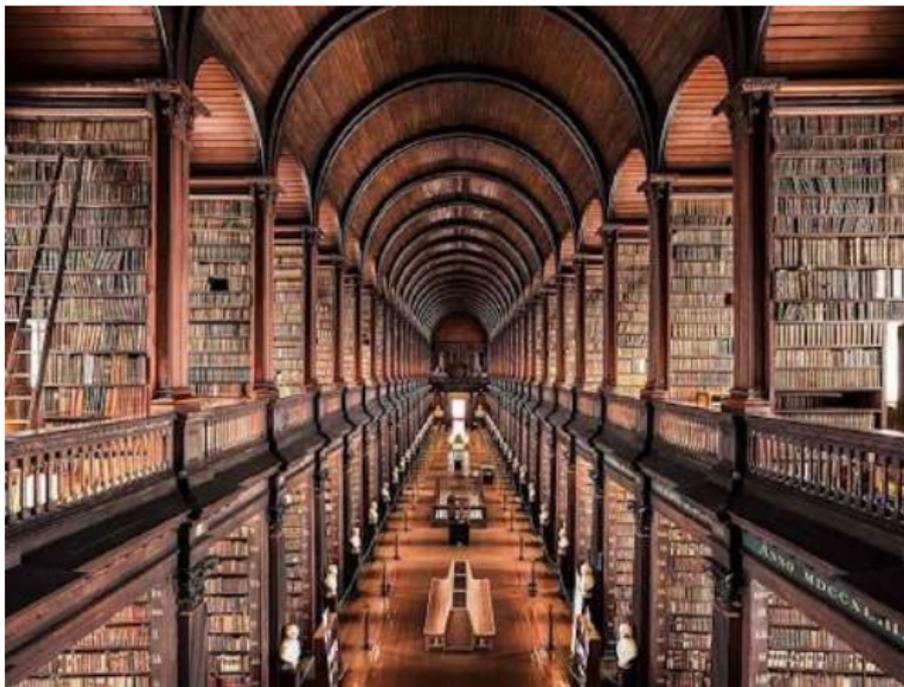- Dictionary of K vectors (or snapshots),

$$\mathcal{D}_K = \{v^{(1)}, \cdots, v^{(K)}\}$$

- Take as library $\mathcal{L}_m^N = \mathcal{L}_m(\mathcal{D}_K)$,

$$\mathcal{L}_m(\mathcal{D}_K) := \Big\{ \sum_{k=1}^{K} x_k v^{(k)} : \ (x_k)_{k=1}^{K} \in \mathbb{R}^K, \ \|x\|_0 \leq m \Big\},$$

Dictionary multi-space: Library

$\mathcal{L}_m(\mathcal{D}_K)$ is **large** $\longrightarrow$ low ε but not fully explorable.

Dictionary multi-space: Selection

- Consider the LASSO$_\alpha$ problem

$$\mathbf{x}^*(\alpha) \in \underset{\mathbf{x} \in \mathbb{R}^K}{\operatorname{argmin}} \frac{1}{2}\|\mathbf{C}\mathbf{x} - \mathbf{w}\|_2^2 + \alpha\|\mathbf{x}\|_1,$$

with $\alpha > 0$, $\mathbf{C} := (\langle w^{(i)}, v^{(j)} \rangle) \in \mathbb{R}^{m \times K}$ and $\mathbf{w} := (\langle w^{(i)}, u \rangle) \in \mathbb{R}^m$.

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- Use LARS$_{\alpha_0}$ to solve LASSO$_{\alpha_0}$ and generate the recoveries

$$\left(V_\alpha\right)_{\alpha \geq \alpha_0}, \quad V_\alpha := \operatorname{span}\{v^{(i)} : \mathbf{x}^*(\alpha)_i \neq 0\}.$$

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- Use $\mathcal{S}$ to select $\alpha_\mathcal{S}$ among this **smaller** family.

Dictionary multi-space: Selection

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Assuming that P_W is injective on $\mathcal{M}$ and that $\mu(\mathcal{M}, W) < \infty$,

$$\|u - A_{\alpha_{\mathcal{S}}}(w)\| \le 2\frac{C}{c}\mu(\mathcal{M}, W) \min_{\alpha > \alpha_0} \|u - A_{\alpha}(w)\|,$$

Parameterized PDEs
Offline-online decomposition for dictionary-based multispace

Parameterized PDEs: Framework

- Assume affinely parameterized operator equation (e.g. PDEs)

$$B(\xi)u = f \quad \text{with} \quad B(\xi) = B^{(0)} + \sum_{q=1}^{m_B} \xi_q B^{(q)}$$

with $B^{(q)} : U \longrightarrow U'$ linear operators.

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with $B^{(q)} : U \longrightarrow U'$ linear operators.

- Assume **uniform control** on the singular values of $B(\xi)$

$$0 < c \le \min_{v \in U} \frac{\|B(\xi)v\|}{\|v\|} \le \max_{v \in U} \frac{\|B(\xi)v\|}{\|v\|} \le C < \infty,$$

Parameterized PDEs: Surrogate distance

- Consider the residual-based distance

$$\mathcal{S}(v, \mathcal{M}) := \min_{\xi \in \mathcal{P}} \|B(\xi)v - f\|, \quad v \in U$$

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$$\mathcal{S}(v, \mathcal{M}) := \min_{\xi \in \mathcal{P}} \|B(\xi)v - f\|, \quad v \in U$$

- $\mathcal{S}(v, \mathcal{M})$ is **computable** by solving a l.s. system with d dofs,

$$\mathcal{S}(v, \mathcal{M}) := \min_{\xi \in \mathcal{P}} \|G(v)\xi - g(v)\|$$

with

$$G(v) := \left(B^{(1)}v \mid \cdots \mid B^{(d)}v \right) \quad \text{and} \quad g(v) := f - B^{(0)}v.$$

Parameterized PDEs: Offline-online decomposition

- Discrete framework $U = \mathbb{R}^{\mathcal{N}}$ with large $\mathcal{N}$.

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- **Offline**: Heavy pre-computations independently on w.

- **Online**: Fast computation of $A_{\alpha_{\mathcal{S}}}(w)$.

Parameterized PDEs: Offline-online decomposition

- Problem: precomputations for $\mathcal{S}$ cost $\mathcal{O}\big(d^2 \, K^2 \, \mathcal{N} \big)$

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- Instead, we use a random matrix $\Theta \in \mathbb{R}^{k \times \mathcal{N}}$ to compute

$$\mathcal{S}^{\Theta}(v, \mathcal{M}) := \min_{\xi \in \mathcal{P}} \|\Theta\big(B(\xi)v - f\big)\| = \min_{\xi \in \mathcal{P}} \|G^{\Theta}(v)\xi - g^{\Theta}(v)\|$$

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Proposition (Derived from [4])

With $k = \mathcal{O}\big(\epsilon^{-2}\ d\ \log(\delta^{-1})\big)$, for any $v \in U$, with probability at least $1 - \delta$ we have

$$\sqrt{1 - \epsilon}\ \mathcal{S}(v, \mathcal{P}) \leq \mathcal{S}^{\Theta}(v, \mathcal{P}) \leq \sqrt{1 + \epsilon}\ \mathcal{S}(v, \mathcal{P}).$$

Parameterized PDEs: Offline

- Write $A_\alpha(w) = \mathbf{U}\mathbf{a}$ with $\mathbf{a} = \mathbf{a}(w) \in \mathbb{R}^{m+K}$ and $\mathbf{U} = (\mathbf{W} \mid \mathbf{V})$.

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- Write the l.s. terms to compute $\mathcal{S}^\Theta$ as

$$G^\Theta(A_\alpha(w)) := \left(\Theta B^{(1)}\mathbf{U}\mathbf{a} \mid \cdots \mid \Theta B^{(d)}\mathbf{U}\mathbf{a}\right)$$

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- Total offline cost (without snapshot computation)

$$\mathcal{O}(mK\mathcal{N} + dK\mathcal{N}\log(\mathcal{N}))$$

Parameterized PDEs: Online

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- **Step 3**: Solve them, each costing $\mathcal{O}(kd^2)$.

- Total online cost with $k = \mathcal{O}(d)$

$$\mathcal{O}(\underbrace{m^2 K}_{\text{LARS}} + \underbrace{md^2 K}_{\text{prepare l.s.}} + \underbrace{d^3 K}_{\text{solve l.s.}}).$$

Numerical Example
Finite Element space $U = \mathcal{H}^1_{\Gamma_D}(\Omega)$ embedded with $\|\nabla \cdot \|_{\mathcal{L}^2(\Omega)}$

Numerical: Thermal block

$$\mathcal{N} \sim 8\,000 \quad \text{and} \quad \begin{cases} -\nabla \cdot \big(\kappa \nabla u\big) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \kappa = \xi_i & \text{in } \Omega_i,\ 1 \le i \le 9, \end{cases}$$
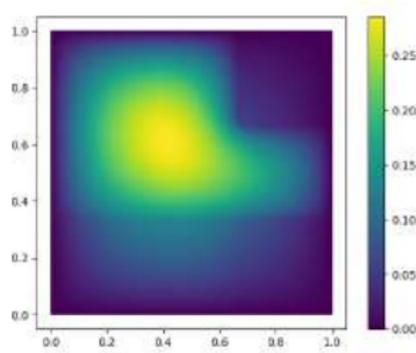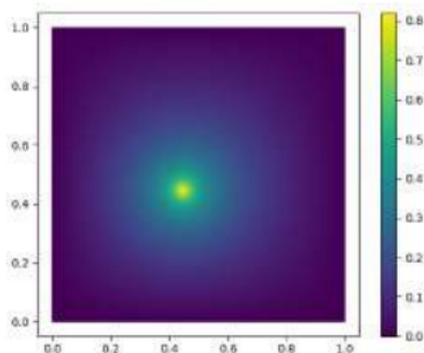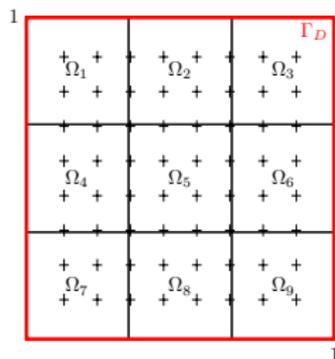

Figure: Left: geometry, with sensors locations (crosses). Middle: Riesz representer of a sensor. Right: a snapshot.
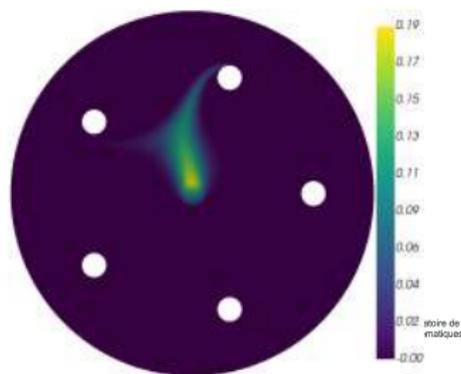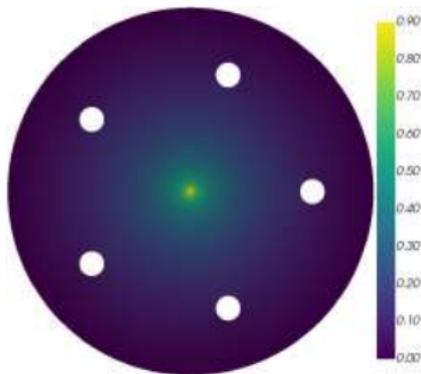
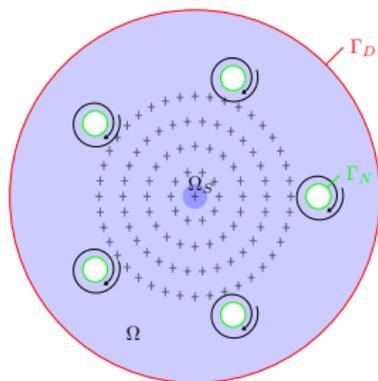## Numerical: Thermal block
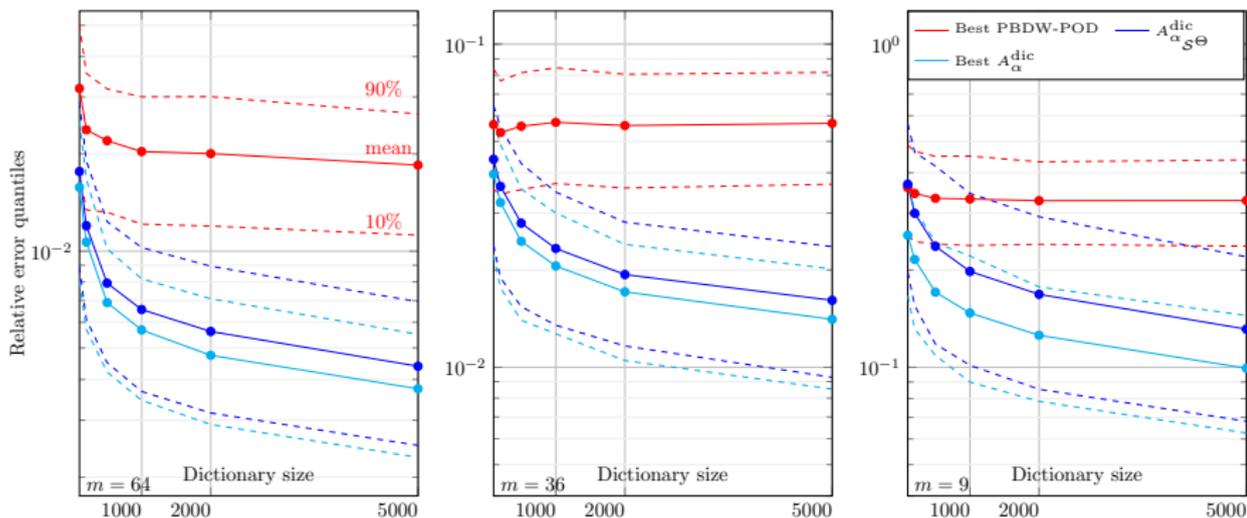
Figure: Evolution of the recovery errors in U-norm, on 500 test snapshots, with growing dictionary sizes.

Numerical: Advection diffusion

$$\mathcal{N} \sim 150\ 000 \quad \text{and} \quad \begin{cases} -0.01\Delta u + \mathcal{V}(\xi) \cdot \nabla u = \dfrac{100}{\pi}\mathbb{1}_{\Omega_S} & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ n \cdot \nabla u = 0 & \text{on } \Gamma_N, \end{cases}$$

$$\mathcal{V}(\xi) = \sum_{i=1}^{5} \frac{1}{\|x - x^{(i)}\|}\left(\xi_i e_r(x^{(i)}) + \xi_{i+5} e_\theta(x^{(i)})\right)$$
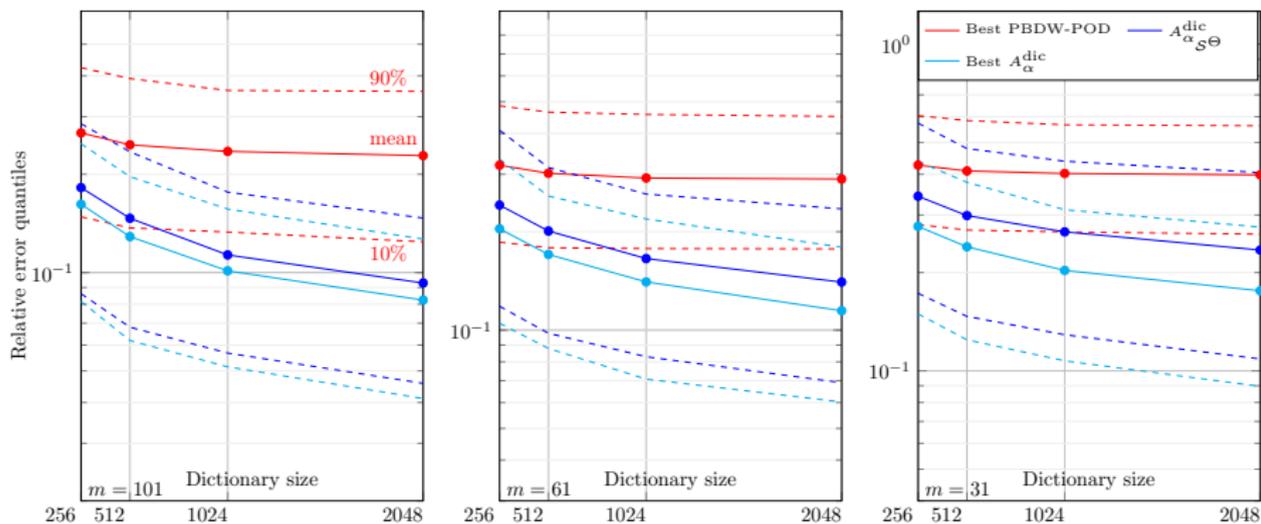
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Conclusion and perspectives

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Perspectives

- Noisy framework ?

- Bi-dictionary approach ?

Thank you !

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