Dictionary-based model reduction for state estimation

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Context

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- Parametric equation in a Hilbert space U,

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$u \in U$ the **state to recover** and $\xi \in \mathcal{P} \subset \mathbb{R}^d$ an **unknown** parameter.

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↓

Compute a recovery $A(w) \simeq u$

One space problem

Linear MOR

One space: formulation

- Approximate $\mathcal{M}$ by a linear subspace V,

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- Parameterized-Background Data-Weak (PBDW) [1]:

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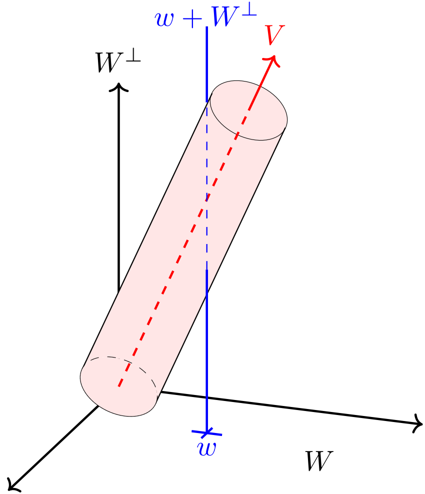
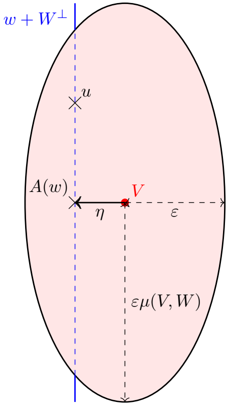
- Sharp error bound [2]: $\|u - A_V(w)\| \leq \mu(V, W)\varepsilon$

$$\mu(V, W) := \beta(V, W)^{-1}, \quad \beta(V, W) := \inf_{v \in V} \sup_{w \in W} \frac{\langle v, w \rangle}{\|v\| \|w\|}$$

One space: Geometry





One space: Pros and cons

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Cons

- Requires $n \leq m$.
- Trade-off between ε and $\mu(V, W)$.
- Limited by the Kolmogorov m-width of $\mathcal{M}$,

$$\varepsilon \geq d_m(\mathcal{M}).$$

Multi-space problem

Library-based MOR

Multi-space: library-based MOR

- Consider a library of spaces $\mathcal{L}_n^N := \{V_1, \cdots, V_N\}$

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- New benchmark: non-linear Kolmogorov (n, N)-width [3]

$$d_n(\mathcal{M}, N) := \inf_{\#\mathcal{L}_n^N = N} \sup_{u\in\mathcal{M}} \min_{V\in\mathcal{L}_n^N} \min_{v\in V} \|u - v\|,$$

which is expected to decay much faster than

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- Aim for low ε with a low n, thus better stability.

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How to select V^* among this library ?

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- Assume we have $\mathcal{S}$ such that for any $v \in U$,

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- Select V^* as

$$V^* \in \underset{V \in \mathcal{L}_n^N}{\operatorname{argmin}} \; \mathcal{S}(A_V(w), \mathcal{M}).$$

Multi-space: Selection

Proposition (Near optimal selection [4])

Assuming that P_W is injective on $\mathcal{M}$ and that $\mu(\mathcal{M}, W) < \infty$,

$$\|u - A_{k^*}(w)\| \leq 2\frac{C}{c}\mu(\mathcal{M}, W) \min_{1\leq k\leq N} \|u - A_k(w)\|,$$

$\mu(\mathcal{M}, W)$ reflects how well $\mathcal{M}$ and W are aligned.

Dictionary-based multi-space

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- Dictionary of K vectors (or snapshots),

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- Take as library $\mathcal{L}_m^N = \mathcal{L}_m(\mathcal{D}_K)$,

$$\mathcal{L}_m(\mathcal{D}_K) := \left\{ \sum_{k=1}^{K} x_k v^{(k)} : \ (x_k)_{k=1}^K \in \mathbb{R}^K, \ \|x\|_0 \leq m \right\},$$

Dictionary multi-space: Library

$\mathcal{L}_m(\mathcal{D}_K)$ is **large** $\longrightarrow$ low ε but not fully explorable.

Dictionary multi-space: Selection

- Consider the LASSO$_\alpha$ problem

$$\mathbf{x}^*(\alpha) \in \underset{\mathbf{x}\in\mathbb{R}^K}{\operatorname{argmin}} \frac{1}{2}\|\mathbf{C}\mathbf{x}-\mathbf{w}\|_2^2 + \alpha\|\mathbf{x}\|_1,$$

with $\alpha > 0$, $\mathbf{C} := (\langle w^{(i)}, v^{(j)}\rangle) \in \mathbb{R}^{m\times K}$ and $\mathbf{w} := (\langle w^{(i)}, u\rangle) \in \mathbb{R}^m$.

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- Use LARS$_{\alpha_0}$ to solve LASSO$_{\alpha_0}$ and generate the recoveries

$$\left(V_\alpha\right)_{\alpha\geq\alpha_0}, \quad V_\alpha := \text{span}\{v^{(i)} : \mathbf{x}^*(\alpha)_i \neq 0\}.$$

with the associated recoveries $A_\alpha(w) := A_{V_\alpha}(w)$

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- Use $\mathcal{S}$ to select $\alpha_{\mathcal{S}}$ among this **smaller** family.

Dictionary multi-space: Selection

Proposition (Near optimal selection [4])

Assuming that P_W is injective on $\mathcal{M}$ and that $\mu(\mathcal{M}, W) < \infty$,

$$\|u - A_{\alpha_S}(w)\| \leq 2\frac{C}{c}\mu(\mathcal{M}, W) \min_{\alpha > \alpha_0} \|u - A_\alpha(w)\|,$$

Parameterized PDEs

Offline-online decomposition for dictionary-based multispace

Parameterized PDEs: Framework

- Assume affinely parameterized operator equation (e.g. PDEs)

$$B(\xi)u = f \quad \text{with} \quad B(\xi) = B^{(0)} + \sum_{q=1}^{m_B} \xi_q B^{(q)}$$

with $B^{(q)} : U \longrightarrow U'$ linear operators.

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- Assume **uniform control** on the singular values of $B(\xi)$

$$0 < c \leq \min_{v \in U} \frac{\|B(\xi)v\|}{\|v\|} \leq \max_{v \in U} \frac{\|B(\xi)v\|}{\|v\|} \leq C < \infty,$$

Parameterized PDEs: Surrogate distance

- Consider the residual-based distance

$$\mathcal{S}(v, \mathcal{M}) := \min_{\xi \in \mathcal{P}} \|B(\xi)v - f\|, \quad v \in U$$

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$$\mathcal{S}(v, \mathcal{M}) := \min_{\xi \in \mathcal{P}} \|B(\xi)v - f\|, \quad v \in U$$

- $\mathcal{S}(v, \mathcal{M})$ is **computable** by solving a l.s. system with d dofs,

$$\mathcal{S}(v, \mathcal{M}) := \min_{\xi \in \mathcal{P}} \|G(v)\xi - g(v)\|$$

with

$$G(v) := \left(B^{(1)}v \mid \cdots \mid B^{(d)}v \right) \quad \text{and} \quad g(v) := f - B^{(0)}v.$$

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- Discrete framework $U = \mathbb{R}^{\mathcal{N}}$ with large $\mathcal{N}$.

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- **Offline**: Heavy pre-computations independently on w.
- **Online**: Fast computation of $A_{\alpha_S}(w)$.

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- Instead, we use a random matrix $\Theta \in \mathbb{R}^{k\times\mathcal{N}}$ to compute

$$\mathcal{S}^\Theta(v,\mathcal{M}) := \min_{\xi\in\mathcal{P}} \|\Theta\big(B(\xi)v - f\big)\| = \min_{\xi\in\mathcal{P}} \|G^\Theta(v)\xi - g^\Theta(v)\|$$

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Proposition (Derived from [5])

With $k = \mathcal{O}\Big(\epsilon^{-2}\ \big(d + \log(\delta^{-1})\big)\Big)$, for any $v \in U$, with probability at least $1 - \delta$ we have

$$\sqrt{1-\epsilon}\ \mathcal{S}(v, \mathcal{P}) \leq \mathcal{S}^\Theta(v, \mathcal{P}) \leq \sqrt{1+\epsilon}\ \mathcal{S}(v, \mathcal{P}).$$

Parameterized PDEs: Offline

- Write $A_\alpha(w) = \mathbf{U}\mathbf{a}$ with $\mathbf{a} = \mathbf{a}(w) \in \mathbb{R}^{m+K}$ and $\mathbf{U} = \big(\mathbf{W} \mid \mathbf{V}\big)$.

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- Write the l.s. terms to compute $\mathcal{S}^\Theta$ as

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- **Offline**: compute $\underbrace{\mathbf{C}}_{\mathcal{O}(mK\mathcal{N})} \in \mathbb{R}^{m\times K}$ and $\underbrace{\Theta B^{(q)}\mathbf{U}}_{\mathcal{O}(K\mathcal{N}\log(\mathcal{N}))} \in \mathbb{R}^{k\times(m+K)}$

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- Total offline cost (without snapshot computation)

$$\mathcal{O}(mK\mathcal{N} \quad + \quad dK\mathcal{N}\log(\mathcal{N}))$$

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- **Step 2**: Prepare the l.s. systems, each costing $\mathcal{O}(kmd)$.
- **Step 3**: Solve them, each costing $\mathcal{O}(kd^2)$.
- Total online cost with $k = \mathcal{O}(d)$

$$\mathcal{O}\Big(\underbrace{m^2K}_{\text{LARS}} + \underbrace{md^2K}_{\text{prepare l.s.}} + \underbrace{d^3K}_{\text{solve l.s.}}\Big).$$

Numerical Example

Finite Element space $U = \mathcal{H}^1_{\Gamma_D}(\Omega)$ embedded with $\|\nabla \cdot\|_{\mathcal{L}^2(\Omega)}$

Numerical: Thermal block

$$\mathcal{N} \sim 8\ 000 \quad \text{and} \quad \begin{cases} -\nabla \cdot (\kappa \nabla u) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \kappa = \xi_i & \text{in } \Omega_i,\ 1 \leq i \leq 9, \end{cases}$$

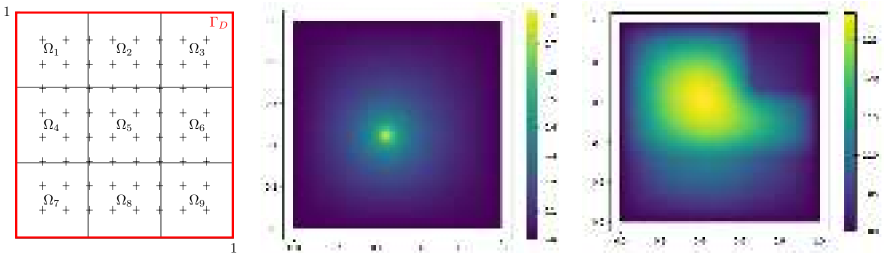

Figure: Left: geometry, with sensors locations (crosses). Middle: Riesz representer of a sensor. Right: a snapshot.

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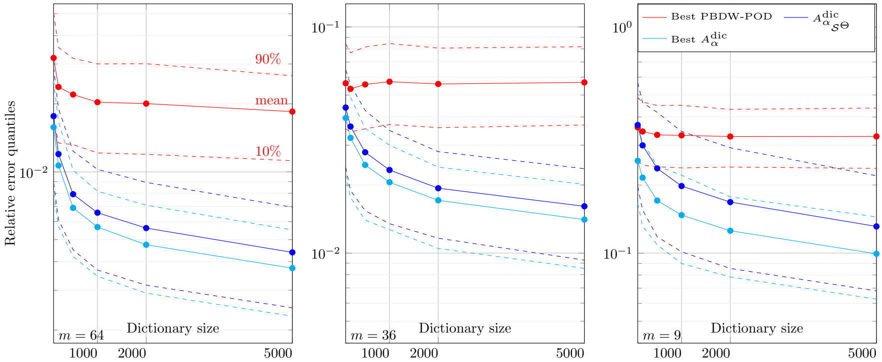
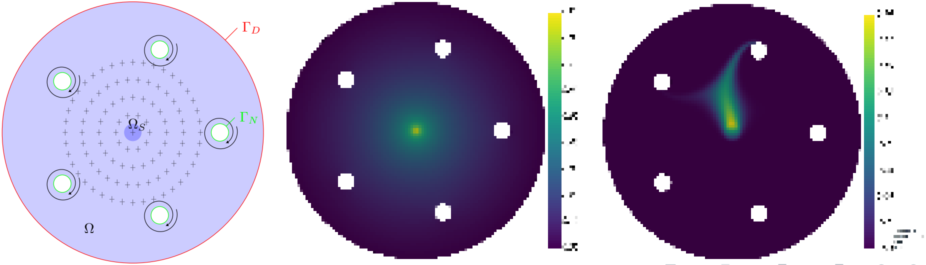


Figure: Evolution of the recovery errors in U-norm, on 500 test snapshots, with growing dictionary sizes.

Numerical: Advection diffusion

$$\mathcal{N} \sim 150\ 000 \quad \text{and} \quad \begin{cases} -0.01\Delta u + \mathcal{V}(\xi)\cdot\nabla u = \dfrac{100}{\pi}\mathbb{1}_{\Omega_S} & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ n\cdot\nabla u = 0 & \text{on } \Gamma_N, \end{cases}$$

$$\mathcal{V}(\xi) = \sum_{i=1}^{5} \frac{1}{\|x - x^{(i)}\|}\left(\xi_i e_r(x^{(i)}) + \xi_{i+5} e_\theta(x^{(i)})\right)$$



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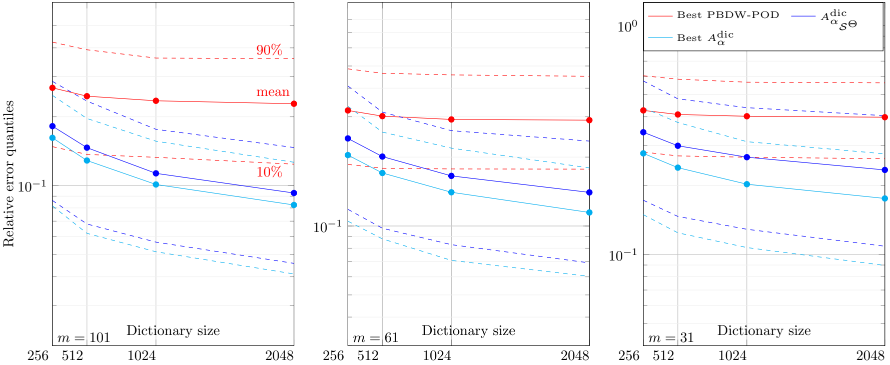

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Conclusion and perspectives

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Perspectives

- Noisy framework ?
- Bi-dictionary approach ?

Thank you !

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